Quantum corrections to finite-gap solutions for Yang-Mills-Nahm equations via zeta-function technique.

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Abstract

One-dimensional Yang-Mills-Nahm models are considered from algebrogeometric points of view. A quasiclassical quantization of the models based on path integral and its zeta function representation in terms of a Green function diagonal for a heat equation with an elliptic potential is considered. The Green function diagonal and, hence, zeta function and its derivative are expressed via solutions of Hermit equation and, alternatively, by means of Its-Matveev formalism in terms of Riemann teta-functions. For the Nahm model, which field is represented via elliptic (lemniscate) integral by construction, one-loop quantum corrections to action are evaluated as the zeta function derivative in zero point in terms of a hyperelliptic integral. The alternative expression should help to link the representations and continue investigation of the Yang-Mills-Nahm models.

Keywords: Nahm model, one-loop quantum corrections, zeta function, elliptic potential, hyperelliptic integral, Its-Matveev formula. MSC numbers: 81Q30, 35J10, 35K08, 81T13.

1 Introduction. On Nahm models

The celebrated Yang-Mills field theory have reductions to one dimensional models [13]. There are two possibilities of such reduction interpretation:

Euclidean version; the equations reduces by deleting dependence on arbitrary three cartesian coordinates, intimately related the Atiyah-Drinfeld-Hitchin-Manin-

Nahm (ADHMN) construction to static monopole solutions to Yang-Mills-Higgs theories in four dimensions in the Bogomolnyi-Prasad-Sommerfield limit. The ADHMN construction supply equivalence between self-dual equations, one - unidimensional, the other in three dimensions (reduced Euclidean four dimensional theory by deleting dependence on a single variable) [1].

Minkowski space: ought to delete dependence on spatial variables: [2].

Let us consider **Yang-Mills equations** in four Euclidean dimensions

$$D_\mu T_{\mu\nu} = 0, \tag{1}$$

for the gauge fields $T_\mu = T_\mu^+,\ \mu = 1,2,3,4$, where

$$T_{\mu\nu} = T_{\nu,\mu} - T_{\mu,\nu} - \imath[T_\mu, T_\nu], \quad D_\mu \Phi = \partial_\mu - \imath[T_\mu, \Phi].$$

Suppose independence on variables x_k, k=1,2,3; setting $x_4 = x$,

$$\tfrac{d^2 T_k}{dx^2} = [T_j[T_j, T_k]], \quad [T_k, \tfrac{dT_k}{dx}] = 0. \tag{2}$$

The self-dual equations,

$$\frac{dT_i}{dx} = \pm \varepsilon_{ijk} T_j T_k, \tag{3}$$

imply Eqs. (2).

This paper develops results of recent publication of the author [3], which mathematical origin strictly relates to the pioneering Its-Matveev paper [4].

In the Sec. 2 we review and specify main definitions of the theory, describing briefly zeta function formalism of quasiclassical approximation for path integral representation of the quantum field amplitudes. The quantum corrections to static solutions of a reduced Nahm model, expressed in terms of elliptic integral are evaluated via Riemann zeta function of heat kernel operator Green function diagonal. The Sec. 3 starts with explicit description of a static solutions of the reduced Nahm model, it continues the formalism description introducing an equation for the Laplace transform of the Green function diagonal. The equation is related to the Hermit equation [9], it is solved algebraically. The inverse Laplace transform is given by hyperelliptic (g=2) integral. The Mellin transformation finalize the zeta function derivation, its derivative give the formula for quantum correction. The fourth section is devoted to Matveev-Its formula application to the spectral problem which appear within the variables division procedure applied to the mentioned Green function problem. Its diagonal values, integrated by spectrum, give alternative expression for the Riemann zeta function and correspondent one-loop corrections. The resulting expression is formed by well-converged theta-functions series which allows its effective numerical evaluation.

2 On quasiclassical quantum corrections in path integral formalism.

2.1 Yang-Mills Lagrangian for the Nahm reduction.

To show more details of the description we restrict ourselves by the case 2x2 matrices T_i (isospin 1/2 case) and scalar field (zero spin) with respect to usual spin classification.

Reductions of Nahm system. There are two possibilities

1. Let σ_i be Pauli matrices, simple substitution $T_i = \phi_i(x)\sigma_i$ gives the Euler system for $\phi_i(x)$ solved in Jacobi functions [5].

2. The ansatz in (3) with upper sign

$$T_i = \phi(x)\alpha_i, i = l, 2, 3, \tag{4}$$

where α_i - constant matrices, yields [1]

$$\begin{array}{c} 2\alpha_i = \sum_{j=1}^{3}[\alpha_j[\alpha_j, \alpha_i]] \\ \phi''(x) = 2\phi^3, \end{array} \tag{5}$$

to be considered in this paper. The equation for $\phi(x)$ enters a specific class (m=0) of Ginzburg-Landau model with the potential function of the field

$$V(\phi) = \phi^4/2 + c. \tag{6}$$

Let us choose the variable x as the proper time and take the Weyl gauge $T_4 = 0$ as in [1], $T_i = \phi(x)\sigma_i$ from (4), then

$$T_{0i} = T_{i,0} = \phi'(x)\sigma_i; \tag{7}$$

more general case of the T_i choice is investigated in [1]. Next, for $i, k = 1, 2, 3$,

$$T_{ik} = T_{i,k} - T_{k,i} - \imath[T_i, T_k] = -\imath\phi^2(x)[\sigma_i, \sigma_k] = 2\phi^2(x)\varepsilon_{ikl}\sigma_l. \tag{8}$$

Hence the Lagrangian density is proportional to (see, e.g. [6])

$$T_{\mu\nu}T^{\mu\nu} = T_{0i}T^{0i} + T_{ik}T^{ik} = (\phi'(x))^2\sigma_i\sigma_i + 4\phi^4(x)\varepsilon_{ikl}\sigma_l\varepsilon_{ikl'}\sigma_{l'} = 3[(\phi'(x))^2 + 8\phi^4(x)]. \tag{9}$$

A quasiclassical quantization is based on the path integral formalism for YM theory [6].

2.2 Quantum corrections via zeta-function

The fields $\phi(x)$ of the classical theory may be considered as stationary or static solutions of the nonlinear Klein-Gordon-Fock equation

$$\phi_{tt} - \phi_{xx} + V''(\phi) = 0,$$

with the operator

$$D = -\partial_x^2 + V''(\phi(x)). \tag{10}$$

The approximate quantum corrections to the solutions equation are obtained by path integral estimated by stationary phase method analog. For a one-dimensional potential $V(\phi(x))$, the operator $D = -\frac{d^2}{dx^2} + u(x)$. appears while the second variational derivative of the action functional (Lagrange density $\mathscr{L} = (\partial\phi)^2/2 - V(\phi)$) is evaluated.

The quantum correction takes the form (for details and refs. see [3])

$$\Delta S_{qu} = S_{qu} - S_{vac} = \frac{\hbar}{2}\ln(\frac{\det D}{\det D_0}). \tag{11}$$

Some comments on vacuum contribution S_{vac} one can find in Appendix.

A link to the diagonal Green function (*heat kernel formalism*) has been used in quantum theory since works by Fock [7]. We study the problem for a periodic u(x), the Green function is defined via:

$$\left(\frac{\partial}{\partial t} + D\right) g_D(t,x,x_0) = \delta(t)\,\delta(x - x_0),$$

where $D = -\frac{d^2}{dx^2} + u(x)$, $u(x) \le u_m$ and $\lambda \in [\lambda_0, +\infty)$. Eventual account of other variables (see again [2]) is shown in Appendix. Let the set $\{\lambda_n\} = S$ be a spectrum of a linear operator D, then the generalized Riemann zeta-function $\zeta_D(s)$ is defined by the equality

$$\zeta_D(s) = \sum_{\lambda_n \in S} \lambda_n^{-s}, \tag{12}$$

as analytic continuation to the complex plane of s from the half plane in which the sum in (12) converges. Differentiating the relation (12) with respect to s at the point $s = 0$ yields

$$\ln(\det D) = \zeta_D'(0). \tag{13}$$

The zeta-function (12) admits the representation via the diagonal of the Green function $g_D(t,x,x_0)$ of the operator $\partial_t + D$.

Define the function

$$\gamma_D(t) = \int g_D(t,x,x)\,dx, \tag{14}$$

which Mellin transformation gives zeta function of the operator D

$$\zeta_D(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_D(t) dt. \tag{15}$$

3 Hermite equation for a Green function diagonal and its solutions

3.1 Static solutions of the reduced Nahm model

Integral of (5)

$$(\phi')^2 = (\phi^2)^2 - b^4 \tag{16}$$

may be considered as the case m=0 of the Ginzburg-Landau model. The fact that the YM Lagrangian is reduced to the form (9) by simple rescaling of ϕ allows to link the mass of a Nahm field particle with the quantum correction to action evaluated below. Due to (16), the solution of Nahm equation is expressed as inversion of the elliptic (lemniscate) integral

$$\int_0^{\phi} \frac{d\phi}{\sqrt{\phi^4 - b^4}} = \frac{1}{b} \int_0^{\frac{\phi}{b}} \frac{dt}{\sqrt{(t^2-1)(t^2+1)}} = x, \tag{17}$$

which gives the Jacobi *sn* function with the imaginary modulus $k = i$

$$\phi = bsn(ibx, i). \tag{18}$$

3.2 Generalized Hermite equation

One can check by direct substitution, that the Laplace transform $\hat{g}_L(p,x,x_0)$ of the Green function diagonal $G(p,x) = \hat{g}_L(p,x,x)$ is a solution of bilinear equation

$$2GG'' - (G')^2 - 4(u(x) - p)G^2 + 1 = 0, \tag{19}$$

which is of the same origin as Hermit equation [9] (see also [10]). In a case of reflectionless and finite-gap solutions the equation (19) is solved more effectively than the equation for the Green function. Namely, the polynomials (in p) P, Q

$$G(p,x) = P(p,)/2\sqrt{Q(p,z)}, \tag{20}$$

where the new variable

$$z = cn^2(bx; i) \tag{21}$$

is introduced (for a compactness of expressions $b \to -ib$ is redefined).

It yields solutions of (19) by plugging (20):

$$b^2(\rho(2PP''-(P')^2)+\rho'PP')-(p+u(z))P^2+Q=0, \tag{22}$$

the primes denote derivatives with respect to z, while account of the definition of $u(x)=6\phi^2$ in (10), explicit expression for V in (6), the form of solution (18) and the new variable expression (21) should have in mind.

$$\begin{array}{c} \rho = z(1-z)(2-z), \\ u(z) = -6b^2\,(1-z)\,. \end{array} \tag{23}$$

In our case of linear $u(z)$, which corresponds the Nahm model (23), the choice is given by

$$\begin{array}{c} P = p^2+P_1(z)p+P_2(z), \\ Q = p^5+q_4p^4+q_3p^3+q_2p^2+q_1p+q_0. \end{array} \tag{24}$$

Plugging (24) into (22) yields (argument z is not shown)

$$\begin{array}{c} -2P_1-u+q_4=0, \\ -2P_2-P_1^2-2uP_1+b^2(2\rho P_1''+\rho'P_1')+q_3=0, \\ b^2(\rho(2P_2''+2P_1P_1''-(P_1')^2)+\rho'(P_2'+P_1P_1'))-2P_1P_2)-u(2P_2+P_1^2)+q_2=0, \\ b^2(2\rho(2P_1''P_2-P_1'P_2'+P_1P_2'')+\rho'(P_1P_2'+P_1'P_2))-P_2^2-2uP_1P_2+q_1=0, \\ b^2(\rho(2P_2P_2''-P_2'^2)+\rho'P_2P_2')-uP_2^2+q_0=0. \end{array} \tag{25}$$

The substitution of (23) into (25) gives the values of the polynomial Q coefficients $q_4=0, \quad q_3=-21b^4, \quad q_2=q_1=108b^8, q_0=0$, hence $P_1(z)=-3b^2(z-1), \quad P_2=18b^4z^2-36b^4z$. Finally,

$$Q=\prod_{i=1}^{i=5}(p-p_i), \tag{26}$$

where the polynomial Q have the simple roots $p_i=\{-2\sqrt{3}b^2,-3b^2,0,3b^2,2\sqrt{3}b^2\}$, ordered for real b^2; obvious reflection symmetry implies an underlying Riemann surface reduction [10].

Let us pick up the expressions determining $\hat{\gamma}(p)$:

$$\hat{\gamma}(p)=\int(p^2-3b^2(z-1)p+18b^4(z^2-2z))dx/2\sqrt{Q}. \tag{27}$$

Going to the variable z and integrating from z=1 to z=0 gives the Laplace transform

$$\hat{\gamma}(p)=[K(\imath)p^2-3b^2(K(\imath)-E(\imath))p-12b^4K(\imath)]/2\sqrt{Q}, \tag{28}$$

As a next step we obtain

$$\gamma(t) = \int \exp[-pt]\hat{\gamma}(p)dp, \tag{29}$$

as inverse Laplace transform. The Mellin transformation gives the zeta function in terms of complete elliptic lemniscate integrals $K(\imath)$ and $E(\imath)$, that are expressed, for example, as hypergeometric series [8].

Plugging the result (28) into the Riemann zeta function (15), denoting for brevity $K(\imath) = K, E(\imath) = E$, yields

$$\zeta(s) = -\int_l \frac{1}{(-p)^s} \frac{2Kp^2+3b^2(K-E)p-48b^4K}{2\sqrt{p(p+3b^2)(p-3b^2)(p-2\sqrt{3}b^2)(2\sqrt{3}b^2+p)}}dp. \tag{30}$$

The change of the variable of integration $p = 3p'b^2, dp = 3dp'b^2$ gives

$$\zeta(s) = \frac{b}{2}\int_l \frac{4K - 3p^2K + 3p[K-E]}{(-3b^2p)^s\sqrt{p(3p^4-7p^2+4)}}ds. \tag{31}$$

Differentiation by s and going to the limit $s \to 0$, gives the integral that is proportional to the mass correction (Nahm particle mass):

$$\zeta'(0) = \frac{b}{2}\int_l \frac{4K - 3p^2K + 3p[K-E]}{\sqrt{p(3p^4-7p^2+4)}} \ln\left(-3b^2p\right) ds. \tag{32}$$

The integral (32) corresponds the hyperelliptic curve of genus 2

$$\mu^2 = p(p-1)(p+1)(p-2/\sqrt{3})(p+2/\sqrt{3}) \tag{33}$$

that is a particular case $\alpha = -1, \beta = 2/\sqrt{3}$ of an example from Sec. 7.1 of the book [10]. The important property of the curve, the presence of non-trivial authomorphism T is described:

$$T : (\mu, p) \to (\frac{\mu(-\beta^{3/2})}{p^3}, \frac{-\beta}{p}). \tag{34}$$

As it is shown in the mentioned book the properties of the curve (33) allow to transform the integral by hyperelliptic curve to the combination of elliptic ones by the correspondent curves.

4 Alternative construction. The generalized zeta-function via Its-Matveev formula.

The one-dimensional Green function in the heat kernel formalism is defined by

$$\left(\frac{\partial}{\partial t} - \frac{d^2}{dx^2} + V''\right) G(t,x,x_0) = \delta(t)\,\delta(x-x_0) \tag{35}$$

with a potential $V''(\phi(x)) = 6b^2 sn^2(\imath bx, \imath)$ originated from a model like (5), see also (18). The function $G(t,x,x_0)$ is supposed to be continuous at $t \geq 0$. The division of variables

$$G(t,x,x_0) = \int g_H(x,x_0)\exp[Ht]dH, \tag{36}$$

yields the spectral problem

$$\left(H - \frac{\partial^2}{\partial x^2} + 6b^2 sn^2(\imath bx, \imath)\right)\psi(x,H) = 0, \tag{37}$$

or, by rescaling $y = \imath bx, h = \frac{H}{b^2}$, results in the Jacobi form of Lam$\acute{e}$ equation [8] (n=2, k=i)

$$\left(h + \frac{\partial^2}{\partial y^2} - 6(\imath sn(y,i))^2\right)\psi(y,h) = 0. \tag{38}$$

The Its-Matveev formula [4] reads as

$$\psi\big(y; h(\mathscr{P})\big) = \mathrm{const}(\mathscr{P})\frac{\Theta\big(\mathscr{A}(\mathscr{P}) + yU + D\big)}{\Theta\big(yU + D\big)}e^{\Omega(\mathscr{P})y}. \tag{39}$$

It is expressed in terms of Riemann theta functions and solves the equation (38) for a spectral parameter h with the potential

$$u = -2\frac{d^2}{dy^2}\ln\Theta\big(Uy + D\big) + const. \tag{40}$$

The parameters $\mathscr{P}, U, D, \mathscr{A}, \Omega(\mathscr{P})$ are defined via Abelian integrals fixed by positions of singular points in (26), see Sec. 7.7 [10], where the reduction to Lam$\acute{e}$ potential case (38) is specified. The theta function of the representation is defined, e.g. in the same book [10] via theta-series. The series convergence is rapid, therefore the representation (39) is convenient for numeric evaluation of the integrals in zeta formalism. The link of the potentials for the case of the Sine-Gordon model (Lam$\acute{e}$ equation with n=1) is expressed in explicit form in [3].

The Green function g_h of the spectral Lam$\acute{e}$ problem (38) may be constructed as a product of two independent solutions $\psi_+(y;h)$, and ψ_- of the spectral equation with the same h:

$$g_h(y,y_0) = \frac{1}{W}\left\{ \begin{array}{ll} \psi_+(y;h)\psi_-(y_0;h), & y < y_0 \\ \psi_-(y;h)\psi_+(y_0;h) & y > y_0. \end{array} \right. \tag{41}$$

The Wronskian factor W is chosen to normalize (43) so that fix the jump of the first derivative with respect to x:

$$\lim_{\varepsilon \to 0}\left[\frac{dg_H(x_0+\varepsilon, x_0)}{dx} - \frac{dg_h(H_0-\varepsilon, x_0)}{dx}\right] = -1. \tag{42}$$

The independent solution $\psi_-(y;h)$ may be chosen antisymmetric with respect to the reflection $x \to -x$.

The result (42) is substituted into the integral by the spectrum

$$G(t,x,x_0) = b^2 \int g_h(\imath bx, \imath bx_0) \exp[hb^2] dh. \tag{43}$$

Finally we integrate the diagonal values of the Green function from (43) by the period of a solution obtaining $\gamma_D(t) = \int G(t,x,x)dx$. After that, using the definition of the generalized zeta-function (49) and calculating the derivative at zero point, one arrives at the mass expression that is proportional to the quantum correction (45).

5 Conclusion

Two alternative expressions (31) and one arising from (43) for the zeta function link the Its-Matveev representation and the Laplace transform (hyperelliptic integral) representation. We study both representations of the zeta function because of eventual significance of the result, its cross-verification and, the necessity of further investigation of both ones. Perhaps, there is some interest of such comparison from mathematical point of view, The value of zeta function derivative at zero point (32) allows to evaluate the quantum corrections to energy that may be considered as the YMN particle mass itself (zero value of classical mass) in the framework of the Nahm model as YM reduction, after the choice of the parameter b, that also need additional physical investigation.

Dressing procedure [11, 12] may be applied directly to the spectral problem (37) or to the evolution equation (35) and widen class of solutions. Some such results concern 2+1 case [13, 14]. The YM equations itself is proved to be non-integrable, but the dressing (gauge-Darboux transformation of [14]) and methods outlined in this article are effective in the case of YM reductions as well as for other non-integrable systems as e.g. Landau-Ginzburg one.

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6 Appendix: model implementation in multidimensions and regularization

Let us consider a problem in d dimensions for a Nahm field that still depends only on x but linkwed to the ADHMN construction.

$$D = -\partial_x^2 - \Delta_y + V''(\phi(x)) = D_x - \Delta_y. \tag{44}$$

$y \in R^{d-1}$ stands for a transverse variables. For a regularization a vacuum action S_{vac} is introduced with $D_0 = -\partial_x^2 - \Delta_y$. The quantum correction takes the form

$$\Delta S_{qu} = S_{qu} - S_{vac} = \frac{\hbar}{2} \ln(\frac{\det D}{\det D_0}). \tag{45}$$

Such extraction is used when the limit case of a kink is studied [3].

For the transversal variables contribution one can use the property of multiplicity: *if the operator D is a sum of two differential operators $D = D_1 + D_2$, which depend on different variables, the following equality holds*

$$\gamma_D(t) = \gamma_{D_1}(t)\gamma_{D_2}(t). \tag{46}$$

It follows directly from definition (14) of $\gamma_D(t)$. For the Laplacian Δ_y the function γ is equal to $d-1$-dimensional Poisson integral

$$\gamma_{D_y}(t) = \frac{1}{(2\pi)^{d-1}} \int_{R^{d-1}} d\mathbf{k} \exp(-|\mathbf{k}|^2 t) = (4\pi t)^{-\frac{d-1}{2}}. \tag{47}$$

Similarly the ”vacuum” one is evaluated. Substitution the expression (47) having in mind (29) for $\gamma(t)$

$$\zeta_D(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left(\gamma_{D_0}(t) - (4\pi t)^{-\frac{d-1}{2}} \gamma(t)\right) dt \tag{48}$$

yields the regularized zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} (\gamma_{D_0}(t) - \gamma_D(t))\, dt. \tag{49}$$

Finally, the quantum correction to action is proportional to

$$\Delta S = \frac{\hbar}{2} \zeta'(0). \tag{50}$$